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**INFRARED BEHAVIOUR OF MASSLESS INTEGRABLE FLOWS
 ENTERING THE MINIMAL MODELS FROM ϕ_{31}**

G. Feverati, E. Quattrini* and F. Ravanini

*Sezione I.N.F.N. e Dip. di Fisica - Univ. di Bologna
 Via Irnerio 46, I-40126 BOLOGNA, Italy*

Abstract

It is known that any minimal model M_p receives along its ϕ_{31} irrelevant direction *two* massless integrable flows: one from M_{p+1} perturbed by ϕ_{13} , the other from Z_{p-1} parafermionic model perturbed by its generating parafermion field. By comparing Thermodynamic Bethe Ansatz data and “predictions” of infrared Conformal Perturbation Theory we show that these two flows are received by M_p with opposite coupling constants of the ϕ_{31} irrelevant perturbation. Some comments on the massless S-matrices of these two flows are added.

*Present address (until April 16, 1996): Physikalisches Institut der Universität Bonn, Nußallee 12, D-5300 Bonn, Germany
 E-mail: feverati@bo.infn.it, quattrini@bo.infn.it, ravanini@bo.infn.it

1 Introduction

Many two-dimensional massless Quantum Field Theories can be seen as Renormalization Group (RG) flows connecting an ultraviolet (UV) Conformal Field Theory (CFT) to an infrared (IR) one [1]. When an infinite number of charges is conserved the theory is integrable. In this case a powerful method of investigation consists in the Thermodynamic Bethe Ansatz (TBA) [2], that can be thought as a set of non-linear coupled integrable equations driving the RG evolution of various quantities along the flow. In principle the TBA equations could be deduced from the (massive or massless) S-matrix of the theory, but the derivation is technically easy only in the case of *diagonal* S-matrices. If a kink structure connecting different colored vacua appears, one must resort to very complicated Bethe Ansatz techniques to diagonalize the color transfer matrix and deduce the TBA system. In spite of this, a lot of TBA systems attached to many integrable theories have been conjectured, and extensively checked against UV perturbative results. In the case of massless theories, however, also the IR limit is given by a non-trivial CFT to which the RG flows is attracted by an irrelevant integrable operator $\Phi(x)$. This simplified sentence means that one can write an IR-effective action approximating the behaviour of the theory at large scales

$$\mathcal{A} = \mathcal{A}_{IR} + g \int d^2x \Phi(x) + \text{h.o.t.} \quad (1)$$

where \mathcal{A}_{IR} (formally) represents the action of the IR CFT. The “h.o.t.” means higher order terms, i.e. an infinite number of higher dimension operators that in principle can contribute as counterterms each one with its independent coupling constant. In other words, this effective theory is not renormalizable. This makes the IR perturbation theory much more difficult than the UV one. In these two-dimensional theories, experience tells us that the UV perturbative series usually has a finite radius of convergence. Instead, the IR one is at best an asymptotic series. In spite of these difficulties we shall see in this letter that the first few orders of the IR series can be reasonably controlled and turn out to be of interest when compared with the *exact* results coming from the integration of the TBA equations. This allows further checks of the validity of TBA as well as calculation of quantities that can shed new insight on the structure of the space of RG flows, as we shall see below.

The idea of comparing TBA results with IR perturbation theory traces back to Al.Zamolodchikov [3, 4], but only the case of $T\bar{T}$ perturbations (T being the stress-energy tensor) were developed enough. Klassen and Melzer [5] went a bit further, by exploring the simplest cases of ϕ_{31} IR perturbations of minimal models. Much of the inspiration of the present work comes from their approach. IR perturbations have been dealt with in a more systematic way, although in a rather different problem, by

Berkovich [6]. Also this paper contains elements that have been illuminating for our analysis.

In this letter we make use of IR perturbation theory to compare the TBA results of two celebrated examples of integrable massless field theories, maybe the most studied ones:

1. the minimal models M_{p+1} perturbed by ϕ_{13} that notoriously flow to M_p . Following [4] we denote these massless theories by $\mathcal{MA}_{p+1}^{(+)}$.
2. the \mathbf{Z}_{p-1} -parafermionic theories perturbed by their $\Psi_1 = \psi_1 \bar{\psi}_1^+ \psi_1^\dagger \bar{\psi}_1^\dagger$ operator (ψ_1 being the generating parafermion), which also are known to flow down to M_p . Following [7] they will be denoted in this paper by $H_{p-1}^{(\pi)}$.

That the minimal models and the parafermion theories have very strict relation in general, even at the pure conformal level, is not a new surprise and can be traced back to the fact that both heavily involve the $A_1^{(1)}$ affine algebra in the deep of their constructions. The interesting fact here is that both theories flow down to M_p “attracted” by the same operator ϕ_{31} . One can wonder which is the feature of the IR ϕ_{31} perturbation theory that distinguishes the two flows. Our investigation has the aim of clarifying this issue and add a new piece of information to the beautiful puzzle of understanding the map of 2 dimensional integrable flows.

2 TBA equations for $\mathcal{MA}_p^{(+)}$ and $H_p^{(\pi)}$ models

A set of TBA equations for $\mathcal{MA}_p^{(+)}$ has been proposed by Al.Zamolodchikov [4], and checked against various tests that give more than reasonable confidence to its correctness. Basically the prediction for the scaling central charge $c(r)$ ($r = MR$ is an adimensional quantity parametrizing the RG flow in terms of the mass scale M and inverse “temperature” R) is

$$c(r) = \frac{3}{\pi^2} \int_{-\infty}^{+\infty} \sum_a d\theta \nu_a(\theta) L_a(\theta) \quad (2)$$

where $\nu_a(\theta) = \frac{r}{2} e^\theta \delta_{a,1} + \frac{r}{2} e^{-\theta} \delta_{a,p-2}$, the index a running from 1 to $p-2$ along a A_{p-2} Dynkin diagram. The $L_a(\theta) = \log(1 + e^{-\varepsilon_a(\theta)})$ are evaluated as solutions of the set of integral equations

$$\nu_a(\theta) = \varepsilon_a(\theta) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\theta' \varphi(\theta - \theta') \sum_b l_{ab} L_b(\theta') \quad (3)$$

with kernel $\varphi(\theta) = 1/\cosh(\theta)$ and l_{ab} incidence matrix of the A_{p-2} diagram. With this TBA one can test even for low p the validity of the results on the $\mathcal{MA}_p^{(+)}$ models established perturbatively by A.Zamolodchikov for high p .

A similar TBA set has been proposed by Fateev and Al. Zamolodchikov [7] to describe, or even better, to give the first evidence, of highly *nonperturbative* flows between parafermions and minimal models, namely the $H_p^{(\pi)}$ theories briefly described above. Again, the TBA equations and the formula for the scaling central charge are given by eqs.(2) and (3), with the same kernel, the only difference now consisting in a different l_{ab} incidence matrix, namely that of the D_p Dynkin diagram.

3 IR perturbation theory

The discussion of IR perturbative series is basically plagued by the existence of infinitely many irrelevant operators $O_j(x)$ that can contribute to it each one with its independent coupling g_j , thus making the theory not renormalizable. However, conformal perturbation theory has the advantage to be strongly constrained by the fact that the operators appearing in the correlators in the perturbative series must be found among the operator content of the unperturbed CFT itself. The effective perturbed theory is defined by the action

$$\mathcal{A} = \mathcal{A}_{M_p} + \int d^2x \sum_{j \in \mathcal{V}} g_j O_j(x) \quad (4)$$

We use the following requirements to restrict the space of IR perturbing operators \mathcal{V}

- all perturbing operators must be *sca*rs, i.e. $\Delta_j = \bar{\Delta}_j$ in order to keep 1+1 Lorentz invariance
- all perturbing operators must be irrelevant, $\Delta_j > 1$
- total derivative operators do not contribute to the action

Even after these restrictions, the space \mathcal{V} is still extremely large and difficult to analyze. But there is another issue which is the main characteristic of these perturbations: their *integrability*. In other words, we know that these perturbations describe, near IR, flows having an infinite number of conserved currents. In particular it is known that both series of models $MA_p^{(+)}$ and $H_p^{(\pi)}$ have for all p , an infinite set of local integrals of motions (IM) of spin 1 mod 2. This does not exclude for some specific models in these series existence of *other* IM's, but at least this series of IM's must be kept conserved all along the flow. All operators destroying even only one of these currents must be discarded from \mathcal{V} . This is the really powerful constraint we can put on our series. It is indeed a fact stressed by many authors that only operators from the families of [1], $[\phi_{13}]$ and $[\phi_{31}]$ have the nice property to preserve all the IM's of spins 1 mod 2. Therefore our space \mathcal{V} can be restricted to these 3 families in the following analysis.

If R denotes the radius of the cylinder where the theory is put for TBA analysis, the scaling function (normalized as to give the UV and IR central charges in the two limits $R \rightarrow 0$ and $R \rightarrow \infty$ respectively) is given near IR by a series expansion in the correlation functions of the $O_j(x)$ operators on the cylinder

$$c(r) = c_{IR} + \sum_{n=1}^{\infty} \sum_{j_1, \dots, j_n \in \mathcal{V}} P_{j_1, \dots, j_n} g_{j_1} \dots g_{j_n} R^{\sum_{i=1}^n y_{j_i}} \quad (5)$$

where

$$P_{j_1, \dots, j_n} = 12 \frac{(-1)^n}{n!} R^{2 - \sum_{i=1}^n y_{j_i}} \int \prod_{i=2}^n d^2 u_i \langle O_{j_1}(0) \prod_{i=2}^n O_{j_i}(u_i) \rangle_{cyl} \quad (6)$$

The coupling constants g_i carry a scale dimension $y_i = 2(1 - \Delta_i)$ each, and as in the usual UV case, there will be a relation with the mass scale M of the form

$$g_i = \kappa_i M^{y_i} \quad (7)$$

where κ_i are numerical coefficients. Comparison with TBA data can be done by first ordering the $\sum_{i=1}^n y_{j_i} = y_a$ in decreasing order (all y_i are negative for irrelevant operators). Define

$$c_a = \sum_{J_a} P_{j_1, \dots, j_n} \kappa_{j_1} \dots \kappa_{j_n} \quad (8)$$

where the set J_a is formed by all the possible choices of j_1, \dots, j_n such that $\sum_{i=1}^n y_{j_i} = y_a$. then the scaling function $c(r)$ has the expansion

$$c(r) = c_{IR} + \sum_a c_a r^{y_a} \quad (9)$$

that can be directly compared with the numerical integration of the TBA equations. What in practice one has to do is to choose a truncation of this series and try to fit it against the TBA numerical data.

In order to explicitly compute the P_{j_1, \dots, j_n} one has to transform them from the cylinder to the sphere. Note that in the case of secondary operators the transformation can be more complicate than the usual one for primary operators considered in [8] to give the “sphere” formulae for P ’s. Luckily, as we shall see, for the purposes of this paper, the only secondary operator we have to consider is the $T\bar{T}$ secondary of the identity. The well-known transformation of the stress-energy tensor creates a term $\frac{c}{24}$ that allows a contribution from the 1-point function of this operator on the cylinder. One must keep in mind this exception in the following analysis. No secondary operator of families different from [1] can have a non-zero cylinder 1-point function.

In writing down the asymptotic series (9) one has to analyze which correlation functions really give a contribution and to which order. Here we give a brief summary

of this analysis in the case of the three families of integrability preserving operators mentioned above.

The [1] family contains as first (scalar) secondary the determinant of the stress-energy tensor $T\bar{T}$. Its renormalization group eigenvalue $y = 2(1 - \Delta)$ is -2 , therefore we expect contribution from its 1-point function at r^{-2} in the IR series. Its 2-point function will contribute at r^{-4} , etc... Next operator in the identity family is : $T^2\bar{T}^2$: whose first contribution appears at order r^{-6} .

The ϕ_{31} operator is the dominating one in the IR series. The first contribution comes from its 2-point function $\langle\phi_{31}\phi_{31}\rangle$ and, as $\Delta_{31} = 1 + \frac{2}{p}$, which implies $y = -4/p$, it is at order $r^{-8/p}$. Next contribution is given by the 3-point function at $r^{-12/p}$. This will be the crucial point of our analysis, as we shall see. Further n-point functions of ϕ_{31} appear at orders $r^{-4n/p}$. Secondaries of ϕ_{31} can give contributions at orders $r^{-4-8/p}$ or higher, as it appears from analysis of the “lowest” case, namely $\langle\phi_{31}\phi_{31}^{(2)}\rangle$. (Here $\phi_{31}^{(2)}$ denotes a generic secondary of ϕ_{31} at level 2). The ϕ_{31} expansion is unaffected by its secondaries if we do not consider terms $O(r^{-4})$. Another interference that can appear is between ϕ_{31} and [1] family. The first term comes from $\langle\phi_{31}\phi_{31}T\bar{T}\rangle$, this is at order $r^{-2-8/p}$, and if we consider only terms of order less than this we are also free of this problem.

Let us now turn to the contributions from the pure $[\phi_{13}]$ family. As ϕ_{13} is relevant, the dominant field of this family allowed to be in \mathcal{V} is $L_{-2}\bar{L}_{-2}\phi_{13}$. Its 2-point function contributes at order $r^{-6+8/(p+1)}$, in any case higher than the limit we already imposed to eliminate interference between ϕ_{31} and $T\bar{T}$. All other correlators of secondaries of ϕ_{13} contribute at order higher than this. Insertions of secondaries of the identity into such correlators raise the order even more. Finally the interferences between secondaries of ϕ_{13} and ϕ_{31} also happen at higher orders, because the 2 and 3-point functions involving both ϕ_{13} and ϕ_{31} are all zero.

We conclude that, if we are content to consider contributions up to $r^{-2-8/p}$ excluded, we can only consider the two operators ϕ_{31} and $T\bar{T}$ in the perturbation expansion. Higher orders not only become more complicate from the “theoretical” point of view, but a careful numerical analysis shows that their contributions begin, for a typical numerical sample of TBA data in the range $200 < r < 2000$, to be in the region where the delicate equilibrium of asymptotical convergence starts to break. Therefore we consider in the following the effective action

$$\mathcal{A} = \mathcal{A}_{IR} + \int d^2x(g\phi_{31} + tT\bar{T}) \quad (10)$$

To finish this section, we have to mention another delicate issue that appears in the perturbative series. When some operator is such that for some integer N it happens that $Ny = -2$, the corresponding N -point function is divergent, even after analytic

IR	model	$c(r) - c_{IR}$
M_5	$\mathcal{M}A_6^{(+)}$	$0.57582(7)r^{-\frac{8}{5}} - 1.340(6)r^{-2} + 0.949(5)r^{-\frac{12}{5}}$
M_5	$H_4^{(\pi)}$	$0.57582(2)r^{-\frac{8}{5}} - 0.000(0)r^{-2} - 0.946(1)r^{-\frac{12}{5}}$
M_6	$\mathcal{M}A_7^{(+)}$	$0.22691(2)r^{-\frac{4}{3}} + 0.206(6)r^{-2} + 0.3268(1)r^{-2}\log r$
M_6	$H_5^{(\pi)}$	$0.22690(8)r^{-\frac{4}{3}} - 0.207(1)r^{-2} - 0.3268(8)r^{-2}\log r$
M_7	$\mathcal{M}A_8^{(+)}$	$0.11938(7)r^{-\frac{8}{7}} - 0.841(4)r^{-\frac{12}{7}} + 1.66(3)r^{-2} - 1.17(6)r^{-\frac{16}{7}}$
M_7	$H_6^{(\pi)}$	$0.11938(9)r^{-\frac{8}{7}} + 0.842(7)r^{-\frac{12}{7}} + 0.00(3)r^{-2} - 1.17(9)r^{-\frac{16}{7}}$
M_8	$\mathcal{M}A_9^{(+)}$	$0.07203(0)r^{-1} - 0.350(0)r^{-\frac{3}{2}} - 0.15(4)r^{-2} - 0.27(7)r^{-2}\log r + 0.5(5)r^{-\frac{5}{2}}$
M_8	$H_7^{(\pi)}$	$0.07203(1)r^{-1} + 0.350(2)r^{-\frac{3}{2}} - 0.17(9)r^{-2} + 0.28(1)r^{-2}\log r - 0.5(1)r^{-\frac{5}{2}}$
M_9	$\mathcal{M}A_{10}^{(+)}$	$0.04730(5)r^{-\frac{8}{9}} - 0.194(3)r^{-\frac{4}{3}} + 0.98(0)r^{-\frac{16}{9}} - 1.8(0)r^{-2} + 1.2(4)r^{-\frac{20}{9}}$
M_9	$H_8^{(\pi)}$	$0.04730(6)r^{-\frac{8}{9}} + 0.194(3)r^{-\frac{4}{3}} + 0.98(6)r^{-\frac{16}{9}} - 0.0(1)r^{-2} - 1.2(5)r^{-\frac{20}{9}}$

Table 1: The numerical fits of TBA data against IR perturbative series

continuation in y . To avoid this singularity, one should introduce a counterterm that results in the appearance of a logarithmic term of the form $r^{-2}\log r$ that must be taken into account when fitting with numerical data. This happens in the model M_p with p even for the $\frac{p}{2}$ -point function of ϕ_{31} . Therefore, in the M_4 model this problem arises already at the first term, i.e. the 2-point function of ϕ_{31} . Similarly, in the M_6 case the 3-point function (which is the crucial one in our analysis) is plagued by this log-term. However, for higher M_p with even p , this interference moves to higher and higher ϕ_{31} correlators, and leaves the 2 and 3 point functions free of this problem, allowing clean application of our argument.

4 Numerical analysis

We have performed a high precision (14 significant digits) numerical integration of the TBA equations in the range $200 < r < 2000$ step 10. Stability of the results has been checked against increasing the number of iterations and decreasing the integration step in θ . The data in this range have been fitted with a function of r^{-1} at the powers predicted by the IR perturbation analysis done in the previous section, truncated at around $r^{-2.5}$. The reason for this choice is explained a little below. The coefficient of the fitted expansion thus found are reported in Tab.1.

We started our analysis with the M_5 IR theory. The M_4 case presents the mentioned anomaly to have the first coefficient divergent, which results in a logarithmic contribu-

IR	model	value of κ	theor. coeff. of $r^{-\frac{12}{p}}$
M_5	$\mathcal{MA}_6^{(+)}$	0.048768029	0.946230
M_5	$H_4^{(\pi)}$	0.048767817	-0.946228
M_6	$\mathcal{MA}_7^{(+)}$	0.046196485	∞
M_6	$H_5^{(\pi)}$	0.046196078	∞
M_7	$\mathcal{MA}_8^{(+)}$	0.042986482	-0.842248
M_7	$H_6^{(\pi)}$	0.042986122	0.842280
M_8	$\mathcal{MA}_9^{(+)}$	0.039903430	-0.350708
M_8	$H_7^{(\pi)}$	0.039903708	0.350732
M_9	$\mathcal{MA}_{10}^{(+)}$	0.037120733	-0.194105
M_9	$H_8^{(\pi)}$	0.037121126	0.194123

Table 2: The theoretical values of the coefficient determined by the 3-point function of ϕ_{31} . The values of κ needed for this calculation are deduced from the coefficient of the 2-point function term.

tion. On the other hand, it has been extensively discussed by Klassen and Melzer in [9]. They give arguments to support the hypothesis that the two flows from M_5 and Z_3 come into M_4 from opposite values of the ϕ_{31} coupling g . Log-terms are totally absent in the next model M_5 . The dominating coefficient comes from the 2-point function of ϕ_{31} . It can be evaluated theoretically by resorting, e.g. to formula (2.28) of [5], which is valid also for the irrelevant ϕ_{31} because it is a primary field. Out of that formula we extract the coefficient κ relating the ϕ_{13} coupling g to the mass scale, $|g| = \kappa M^{-4/p}$. Then use of this κ and of the structure constant $C_{(31),(31)}^{(31)}$ that one can get from [10] allows to compute the “theoretical” value of the coefficient of the 3-point function. The only case where this does not work is M_6 where the 3-point function of ϕ_{31} diverges leading to appearance of a log-term and also interferes with $T\bar{T}$. In all other cases we are able to report the theoretical coefficient that we use as a check of the numerical correctness of our data. The most important thing that one immediately sees from Tab.1 is that the coefficient of the 3-point function of ϕ_{31} in the $\mathcal{MA}_{p+1}^{(+)}$ model is always opposite to that of the $H_{p-1}^{(\pi)}$ one, a clear sign that **the two flows enter the IR model from opposite directions!** This definitely settles the problem of the two ϕ_{31} attracted flows that enter each minimal model. But a more careful analysis of the numerical data of Tab.1 show other interesting surprises. For example, within numerical error it seems plausible to take, at least in all models with *odd* M_p as IR, the coupling with $T\bar{T}$ equal to zero for the parafermionic flows, while it is different from zero for the minimal models ones. If

this fact has a relation with the different forms (with or without θ shift – see below) of the two massless S-matrices of the models is an open and interesting issue. For even p instead, it happens that the $T\bar{T}$ coupling always interferes with some n -point ϕ_{31} function. It is curious to note that the renormalized couplings still show the desired behaviour (equal absolute value and opposite/equal signs for odd/even n).

Our numerical data seem to be reliable up to some $r^{-2.5}$. Introduction of terms higher than this do not seem to improve the fit, in some cases even they make convergence worse. This is a typical behaviour of asymptotic series, adding new terms can give worse approximations of the function after a certain point. It could be possible that this $r^{-2.5}$ plays a role of a limit after which asymptotic convergence becomes more problematic. Luckily, this also is the limit above which secondaries, interferences between different families and other such phenomena are confined (at least for $p < 16$), a lucky situation indeed allowing us to observe the sign change in g so important from the physical point of view, before it becomes hidden by all these more or less uncontrollable contributions. Note that number of ϕ_{31} correlators which fall before this limit *increases* with p . Curiously, higher minimal models allow a better IR analysis than the lower ones. Indeed starting from M_7 we are able to see also the 4 and 5-point functions. The 4-point function, as expected, gives the same coefficient in the two flows, while the 5-point again presents the change of sign, thus confirming once more that the two flows come to M_p from opposite directions.

5 Some comments on the S-matrices

For the two classes of flows under considerations, massless S-matrices can be written. The minimal models case has been analyzed in [11], in relation with the so called “imaginary Sine-Gordon” model of relevance for polymer physics. The $S_{LL} = S_{RR}$ matrix of \mathcal{M}_p is formally equal to the S-matrix of the M_{p-1} model perturbed by ϕ_{13} in the *massive* direction, and therefore it is given by the one described in [12]. The S_{RL} and S_{LR} blocks, instead, turn out to be proportional to the S_{LL} block, calculated for a value of θ shifted by an imaginary quantity α .

$$S_{LR}(\theta) \propto S_{LL}(\theta + i\alpha) \quad , \quad S_{RL}(\theta) \propto S_{LL}(\theta - i\alpha) \quad (11)$$

The shift can be fixed by requiring that the two “mixed” components are equal, which should correspond to requiring time-reversal invariance. This imposes $\alpha = \frac{n\pi}{2\mu}$ ($\mu = \frac{1}{p-1}$), $n = 0, 1, \dots$. As shown in ref. [11] the correct choice for this case is given by $\alpha = \frac{\pi}{2\mu}$. The proportionality factor is fixed by the analytical crossing-unitarity requirement.

For what concerns the $H_p^{(\pi)}$ flow, the massless S matrix can be obtained by the following argument. Fateev [13] wrote the massive S matrix of the $H_p^{(0)}$ theories, i.e. the

\mathbf{Z}_p -parafermionic models perturbed by their fundamental parafermion operator in the *massive* regime. He got the S matrix as quantum group reduction of the S matrix of the *Sausage model* (see ref. FOZ). This latter is basically the spectral parameter dependent R-matrix of the $sl(2)_q$ quantum group in the spin 1 representation (in contrast to the Sine-Gordon S matrix, which is in the spin $\frac{1}{2}$ representation). At q root of 1, the quantum group reduction gives the S matrix of the $H_p^{(0)}$ models, as one can easily check against the known cases $p = 2, 3, 4$. If this procedure works fine in the massive sector, we do not see any reason preventing its extension to the massless regime. The S-matrix for the massless Sausage model is written in [14], and consists of blocks all equal to the Sine-Gordon S-matrix, i.e. $S_{LL} = S_{RR} = S_{RL} = S_{LR} = S_{SG}$. This is not unexpected, if we remember that the IR limit of the massless sausage is the $c = 1$ CFT, that is also the UV limit of Sine-Gordon theory. The q -reductions of this S-matrix describe the parafermionic flows $H_p^{(\pi)}$. This also implies that the S_{LL} and S_{RR} components are equal for the two flows $\mathcal{MA}_{p+1}^{(+)}$ and $H_{p-1}^{(\pi)}$. Again, this is not surprising, as the two flows share the same IR limit M_p . The distinction between the two cases is given by the $S_{LR} = S_{RL}$ blocks that in the $\mathcal{MA}_{p+1}^{(+)}$ case have the shift in θ , eq.(11) while in the $H_{p-1}^{(\pi)}$ case $S_{LR}(\theta) = S_{LL}(\theta)$.

This observation would exhaust the problem of writing the massless S-matrices for the two series of flows examined in this paper. However, one has to check the coincidences $H_2^{(\pi)} = \mathcal{MA}_3^{(+)}$ (Ising model) and $H_3^{(\pi)} = \mathcal{MA}_5^{(+)}$ (3 state Potts Model). While the first one does not give any problem, the Potts case is rather intriguing. The S_{LL} and S_{RR} blocks coincide, as expected, but the $S_{LR} = S_{RL}$ do not. On the other hand, a TBA analysis similar to the one done in [11] (and generalizing the massive case of [3] for the scaling tricritical Ising model) shows that both S matrices give the correct $c_{UV} = \frac{4}{5}$ and $c_{IR} = \frac{7}{10}$ (note that that the S matrix with unshifted S_{LR} giving $c_{UV} = \frac{9}{5}$ in [11] differs from ours by a non-trivial Z-factor). It is evident that there must be a transformation of the asymptotic particle basis relating one S matrix to the other. The most appealing feature in this respect is the fact that the two theories $\mathcal{MA}_5^{(+)}$ and $H_3^{(\pi)}$ are in a sense related one another by an orbifold procedure. Consider their UV limits: the $\mathcal{MA}_5^{(+)}$ is the *diagonal* minimal model M_5 , the $H_3^{(\pi)}$ one is the \mathbf{Z}_3 -parafermion, which coincides with the *complementary* modular invariant $((A_4, D_4)$ in the Cappelli, Itzykson and Zuber classification [15]). One passes from one to the other modular invariant by an orbifold procedure. If we make the hypothesis that also off-criticality the two models are related one another by such an orbifold, the S matrix of the $H_3^{(\pi)}$ model will be obtained from the $\mathcal{MA}_5^{(+)}$ one by the orbifold recipe explained in [16]. This indeed works fine in the massive analogs of these two models, as well as for the $S_{LL} = S_{RR}$ blocks here, which turn out to be equal in the two cases, thanks to the automorphism $A_3 \equiv D_3$ of their

adjacency diagrams. However, straightforward application of this folding procedure to the $S_{LR} = S_{RL}$ blocks would predict also for them equality in the two models, while, as seen, they are definitely *not* equal in this case. Our opinion is that the folding procedure à la Fendley Ginsparg [16] has to be modified slightly in the massless case, in the sense that the role of left and right movers are inverted: if we denote by $-$, 0 , $+$ the 3 nodes of A_3 , the R_+ right mover behaves exactly as L_- left mover in the folding procedure, and so on. The $- \leftrightarrow +$ symmetry of A_3 guarantees that if only right or only left objects are concerned (S_{LL} and S_{RR}) this inversion is uninfluent. If instead we are considering the scattering of a left and a right movers this modifies the folding procedure in such a way to give the shifted S_{LR} from the unshifted one (they are related one another by sending $\sinh \rightarrow i \cosh$ in this case). We intend to return to this curiosity and to the general problem of writing S matrices for non-diagonal ϕ_{13} perturbed minimal models in massive and massless regimes in future.

6 Conclusions

In this letter we have given evidence that the two flows entering any minimal model from ϕ_{31} , namely the integrable theories $\mathcal{MA}_{p+1}^{(+)}$ and $H_{p-1}^{(\pi)}$, come into their IR limit from opposite directions. This result can be interesting in itself, as it sheds further light on the map of the two-dimensional RG space of integrable flows and of minimal models in particular. Further things should be investigated to better understand this issue. We have seen in the previous section that the “fine structure” distinction between different modular invariants may play a role in solving apparent contradictions between S matrices that appear to be different and are expected to be equal, as in the commented $H_3^{(\pi)}$ versus $\mathcal{MA}_5^{(+)}$ case. Also, the flows examined in this paper are not unique: the parafermions themselves have in general more than one modular invariant. It would be interesting to solve the whole puzzle of the flows between all parafermions and minimal models ones. This could also point out, by “undoing the truncation” procedures [11], to some orbifold version of the imaginary Sine-Gordon and of the Sausage models. Work in this direction is in progress.

We think that the most appealing fact of the result of the present paper is the way it has been obtained, i.e. by use of IR perturbation, an instrument used only marginally so far in TBA analysis. It is possible that a better development of this IR techniques, maybe by comparison with the similar ones developed in [6], could provide new tools of investigation of the massless integrable theories.

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